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An extension of the slash-elliptical distribution

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Abstract

This paper introduces an extension of the slash-elliptical distribution. This new distribution is generated as the quotient between two independent random variables, one from the elliptical family (numerator) and the other (denominator) a beta distribution. The resulting slash-elliptical distribution potentially has a larger kurtosis coefficient than the ordinary slash-elliptical distribution. We investigate properties of this distribution such as moments and closed expressions for the density function. Moreover, an extension is proposed for the location scale situation. Likelihood equations are derived for this more general version. Results of a real data application reveal that the proposed model performs well, so that it is a viable alternative to replace models with lesser kurtosis flexibility. We also propose a multivariate extension.

MSC: 60E05.

Keywords: Slash distribution, elliptical distribution, kurtosis.

1. Introduction

A distribution closely related to the normal distribution is the slash distribution. This distribution can be represented as the quotient between two independent random variables, a normal one (numerator) and the power of a uniform distribution (denominator). To be more specific, we say that a random variable S follows a slash distribution if it can be written as

$$S = Z/U^{\frac{1}{q}}, \tag{1}$$

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where $Z \sim N(0,1)$ is independent of $U \sim U(0,1)$ and $q > 0$. For $q = 1$, the standard (canonical) version follows and as $q \to \infty$, the standard normal distribution follows. The density function for the standard slash distribution is then given by

$$p(x) = \begin{cases} \dfrac{\phi(0) - \phi(x)}{x^2} & x \neq 0 \\ \dfrac{1}{2}\phi(0) & x = 0 \end{cases} \tag{2}$$

where ϕ denotes the density function of the standard normal distribution (see Johnson, Kotz and Balakrishnan 1995). This distribution has thicker tails than the normal distribution, that is, it has greater kurtosis. Properties of this distribution are studied in Rogers and Tukey (1972) and Mosteller and Tukey (1977). Maximum likelihood estimation for location and scale parameters is studied in Kafadar (1982). Wang and Genton (2006) develop multivariate symmetric and asymmetric versions of the slash distribution. Gómez, Quintana and Torres (2007) and Gómez and Venegas (2008) propose univariate and multivariate extensions of the slash distribution by replacing the normal distribution by the elliptical family of distributions. Asymmetric versions of this family are discussed in the work of Arslan (2008). Arslan and Genc (2009) discuss a symmetric extension of the multivariate slash distribution and Genc (2007) investigates a symmetric generalization of the slash distribution. Gómez, Olivares-Pacheco and Bolfarine (2009) use the slash-elliptical family to extend the Birnbaum-Saunders (BS) distribution. Finally, Genc (2013) introduces a skew extension of the slash distribution utilizing the beta-normal distribution.

The present paper focuses on extending the slash-elliptical family of distributions considered in Gómez et al. (2007) to a distribution with greater kurtosis, for which purpose it is necessary to replace the uniform distribution by the beta distribution. This gives a family of distributions, containing the slash-elliptical family, with much greater flexibility.

The paper is organized as follows. In Section 2 we present the standard versions of the slash distribution and some of its properties. In Section 3 we propose the extension investigated in the paper, called the extended slash-elliptical family of distributions, and study some of its properties. Section 4, which deals with a real data application, reveals that the extended slash-elliptical distribution can be quite useful in fitting real data and substantially improve less flexible models. Parameter estimation is dealt with by using the maximum likelihood approach. Section 5 introduces a multivariate version of the distribution, and Section 6 presents our main conclusions.

2. Preliminaries

In this section we discuss some properties of the ordinary univariate and multivariate slash distributions, for the sake of notation and comparisons.

We say that a random variable X follows an elliptical slash distribution with location parameter μ and scale parameter σ if its density function is of the form

$$f_X(x;\mu,\sigma) = \frac{1}{\sigma} g\left(\left(\frac{x-\mu}{\sigma}\right)^2\right),$$

for some nonnegative function $g(u)$, $u \geq 0$, such that $\int_0^\infty u^{-1/2} g(u)\,du = 1$. We denote $X \sim E\ell(\mu,\sigma;g)$.

In the multivariate setup, a p-dimensional random vector $\mathbf{Y} = (Y_1,\ldots,Y_p)^\mathsf{T}$ follows an elliptical distribution with location parameter vector $\boldsymbol{\mu}$ and scale parameter matrix $\boldsymbol{\Sigma}$, which is positive definite, if its density function is given by

$$f_\mathbf{Y}(\mathbf{y}) = \boldsymbol{\Sigma}^{-1/2} g^{(p)}\left((\mathbf{y}-\boldsymbol{\mu})^\mathsf{T} \boldsymbol{\Sigma}^{-1} (\mathbf{y}-\boldsymbol{\mu})\right),\ \mathbf{y} \in \mathbb{R}^p$$

where $g^{(p)}$ is the density generator function satisfying

$$\int_0^\infty u^{p-1} g^{(p)}(u^2)du < \infty.$$

We use the notation $\mathbf{Y} \sim El_p(\boldsymbol{\mu},\boldsymbol{\Sigma};g^{(p)})$. If the moments of each element of the random vector $\mathbf{Y}$ are finite, then it follows that $E(\mathbf{Y}) = \boldsymbol{\mu}$ and $Var(\mathbf{Y}) = \alpha_g\,\boldsymbol{\Sigma}$, where α_g is a positive constant, as seen for example, in Fang, Kotz and Ng (1990) and Arellano-Valle, Bolfarine and Vilca-Labra (1996).

An extension of the slash model studied in Gómez et al. (2007), called the slash-elliptical distribution, is defined as

$$X = \frac{Z}{U^{1/q}} \tag{3}$$

where $Z \sim E\ell(0,1;g)$ and $U \sim \mathscr{U}(0,1)$, Z and U are independent random variables with $q > 0$. We use the notation $X \sim SE\ell(0,1,q;g)$. The density function for the random variable $X \sim SE\ell(0,1,q;g)$ is given by

$$f_X(x;0,1,q) = \begin{cases} \dfrac{q}{2|x|^{q+1}} \displaystyle\int_0^{x^2} t^{\frac{q-1}{2}} g(t)\,dt & \text{if} \quad x \neq 0 \\ \dfrac{q}{1+q} g(0) & \text{if} \quad x = 0. \end{cases} \tag{4}$$

A location-scale extension for the slash-elliptical distribution is given by $X = \sigma \frac{Z}{U^{1/q}} + \mu$, so that its density function can be written as

$$f_X(x;\mu,\sigma,q) = \frac{q}{\sigma}\int_0^1 u^q g\left(\left(\left[\frac{x-\mu}{\sigma}\right]u\right)^2\right) du, \tag{5}$$

$-\infty < x < \infty$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $q > 0$. We use the notation $X \sim SE\ell(\mu,\sigma,q;g)$.

3. The extended slash-elliptical distribution and its properties

In this section we consider a stochastic representation, the density function (with some graphical representations) and properties for the extended slash distribution.

3.1. Stochastic representation

The stochastic representation of the new distribution is given as

$$X = \frac{W}{T} \tag{6}$$

where $W \sim E\ell(0,1;g)$ and $T \sim Beta(\alpha,\beta)$ are independent random variables with $\alpha > 0$, $\beta > 0$. We call the distribution of X the extended slash elliptical distribution, and use the notation $X \sim ESE\ell(0,1,\alpha,\beta;g)$.

3.2. Density function

The following result shows that the density function of the random variable $ESE\ell$, can be generated using the stochastic representation in (6).

Proposition 1 *Let $X \sim ESE\ell(0,1,\alpha,\beta;g)$. Then, the density function of X is given by*

$$f_X(x) = \begin{cases} \dfrac{1}{2B(\alpha,\beta)|x|^{\alpha+1}}\displaystyle\int_0^{x^2} g(u)u^{\frac{\alpha-1}{2}}\left(1-\frac{u^{1/2}}{|x|}\right)^{\beta-1} du, & if \quad x \neq 0 \\ \dfrac{\alpha}{\alpha+\beta}g(0), & if \quad x = 0 \end{cases} \tag{7}$$

with $\alpha > 0$, $\beta > 0$, and $g(\cdot)$ density generator function.

Proof. From the stochastic representation (6), we have

$$\begin{aligned} W \sim E\ell(0,1;g) \quad &\Rightarrow \quad f_W(w) = g(w^2) \\ T \sim Beta(\alpha,\beta) \quad &\Rightarrow \quad f_T(t|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}, 0 \leq t \leq 1, \end{aligned}$$

in which

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,$$

which can be written as

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Moreover, using the stochastic representation in (6) and the Jacobian transformation approach, it follows that:

$$\left.\begin{array}{l} X = \dfrac{W}{T} \\ Y = T \end{array}\right\} \Rightarrow \begin{array}{l} W = XY \\ T = Y \end{array} \Rightarrow J = \begin{vmatrix} \dfrac{\partial W}{\partial X} & \dfrac{\partial W}{\partial Y} \\ \dfrac{\partial T}{\partial X} & \dfrac{\partial T}{\partial Y} \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y.$$

Hence,

$$\begin{aligned} f_{X,Y}(x,y) &= |J| f_{W,T}(xy,y) \\ f_{X,Y}(x,y) &= y f_W(xy) f_T(y), -\infty < x < \infty, 0 \leq y \leq 1. \end{aligned}$$

Therefore,

$$\begin{aligned} f_X(x) &= \int_0^1 y f_W(xy) \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy \quad , \quad -\infty < x < \infty \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 f_W(xy) y^{\alpha}(1-y)^{\beta-1}\,dy \quad , \quad -\infty < x < \infty, \end{aligned}$$

with $f_W(w) = g(w^2)$ as the density function of W. Hence,

$$f_X(x) \quad = \quad \frac{1}{B(\alpha,\beta)} \int_0^1 g(x^2y^2) y^{\alpha}(1-y)^{\beta-1}\,dy \quad , \quad -\infty < x < \infty. \tag{8}$$

a) For $x = 0$,

$$
\begin{aligned}
f_X(x) &= \frac{1}{B(\alpha,\beta)} \int_0^1 g(0) y^{\alpha} (1-y)^{\beta-1}\, dy \\
&= g(0) \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \int_0^1 \frac{1}{B(\alpha+1,\beta)} y^{(\alpha+1)-1} (1-y)^{\beta-1}\, dy \\
&= g(0) \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \\
&= g(0) \frac{\alpha}{\alpha+\beta}.
\end{aligned}
$$

b) For $x \neq 0$,

$$
f_X(x) = \frac{1}{B(\alpha,\beta)} \int_0^1 g(x^2 y^2) y^{\alpha} (1-y)^{\beta-1}\, dy.
$$

Furthermore, let

$$
\begin{aligned}
& u = x^2 y^2 \Rightarrow y^2 = \frac{u}{x^2} \Rightarrow y = \frac{u^{1/2}}{|x|} \\
& du = 2x^2 y\, dy,
\end{aligned}
$$

so that

$$
\begin{aligned}
f_X(x) &= \frac{1}{2x^2 B(\alpha,\beta)} \int_0^{x^2} g(u) \left(\frac{u^{1/2}}{|x|} \right)^{\alpha-1} \left(1 - \frac{u^{1/2}}{|x|} \right)^{\beta-1} du \\
&= \frac{1}{2B(\alpha,\beta)|x|^{\alpha+1}} \int_0^{x^2} g(u) u^{\frac{\alpha-1}{2}} \left(1 - \frac{u^{1/2}}{|x|} \right)^{\beta-1} du.
\end{aligned}
$$

Then,

$$
f_X(x) = \begin{cases} \dfrac{1}{2B(\alpha,\beta)|x|^{\alpha+1}} \displaystyle\int_0^{x^2} g(u) u^{\frac{\alpha-1}{2}} \left(1 - \frac{u^{1/2}}{|x|} \right)^{\beta-1} du & if \quad x \neq 0 \\ \dfrac{\alpha}{\alpha+\beta} g(0), & if \quad x = 0, \end{cases}
$$

concluding the proof. □

3.3. Some special cases

We now consider some special important cases that can be obtained from the general distribution of $X \sim ESE\ell(0,1,\alpha,\beta;g)$ presented previously.

Example 1 (Slash-elliptical) If X is distributed according to the extended-slash distribution, then $\beta = 1$ (see Gómez et al., 2007). Hence, the pdf of X, can be shown to be given by

$$f_X(x) = \begin{cases} \dfrac{1}{2B(\alpha,1)|x|^{\alpha+1}} \displaystyle\int_0^{x^2} g(u)u^{\frac{\alpha-1}{2}}\,du, & if \quad x \neq 0 \\ \dfrac{\alpha}{\alpha+1} g(0), & if \quad x = 0. \end{cases} \tag{9}$$

Example 2 (Extended-slash) If X is distributed according to the extended-slash (ES) distribution, then $g(u) = \frac{1}{\sqrt{2\pi}} e^{-u/2}$. Hence, the pdf of X can be shown to be given by

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}B(\alpha,\beta)|x|^{\alpha+1}} \displaystyle\int_0^{x^2} e^{-u/2} u^{\frac{\alpha-1}{2}} \left(1 - \frac{u^{1/2}}{|x|}\right)^{\beta-1} du, & if \quad x \neq 0 \\ \dfrac{\alpha}{\alpha+\beta} g(0), & if \quad x = 0 \end{cases} \tag{10}$$

If $\beta = 1$, then one obtains the slash distribution (see Johnson et al., 1995)

Example 3 (Extended-slash-Student-t) If X is distributed according to the extended-slash distribution, then $g(u) = \frac{\Gamma(\frac{1+\nu}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} (1 + \frac{u}{\nu})^{-\frac{1+\nu}{2}}$. Hence, the pdf of X, is given by

$$f_X(x) = \begin{cases} \dfrac{\Gamma(\frac{1+\nu}{2})}{2\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}B(\alpha,\beta)|x|^{\alpha+1}} \displaystyle\int_0^{x^2} (1 + \frac{u}{\nu})^{-\frac{1+\nu}{2}} u^{\frac{\alpha-1}{2}} \left(1 - \frac{u^{1/2}}{|x|}\right)^{\beta-1} du, & if \quad x \neq 0 \\ \dfrac{\alpha}{\alpha+\beta} g(0), & if \quad x = 0. \end{cases} \tag{11}$$

If $\beta = 1$, then one obtains the slash-Student-t distribution (see Gómez et al. 2007).

In the following we illustrate graphically the behaviour of the density function of the extended slash-elliptical distribution for α fixed and for the normal and Student-t (with 5 degrees of freedom) and density function generators, respectively.

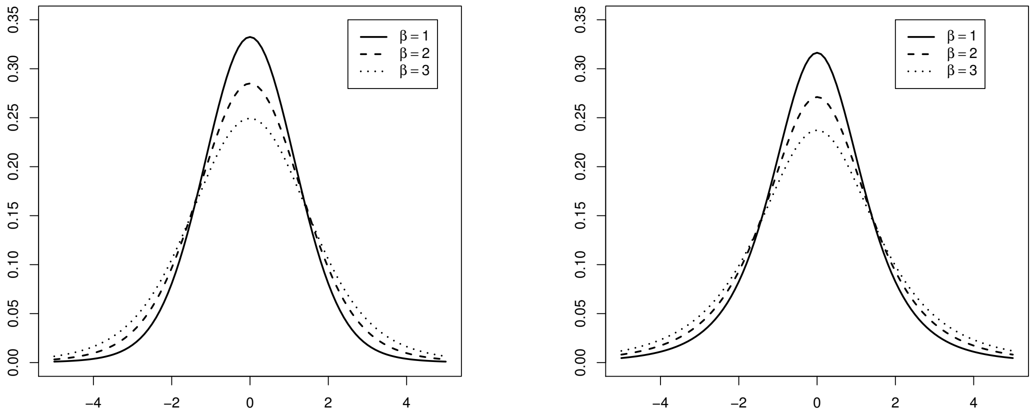

Figure 1: *Density functions for the extended slash distributions with normal density generator (left) and Student-t density generator (right), for $\alpha = 5$ and several values of β.*

3.4. Moments

Proposition 2 *If $X \sim ESE\ell(0,1,\alpha,\beta;g)$, the r-th moment of X is given by*

$$E[X^r] = \frac{\Gamma(\alpha - r)\Gamma(\alpha+\beta)}{\Gamma(\alpha - r + \beta)\Gamma(\alpha)} a_{r/2}, \tag{12}$$

in which

$$a_{r/2} = \int_{-\infty}^{\infty} w^r g(w^2)\, dw. \tag{13}$$

Proof. From the stochastic representation, $X = \frac{W}{T}$, in which $W \sim E\ell(0,1;g)$ and $T \sim Beta(\alpha,\beta)$ are independent random variables, we have

$$E[X^r] = E\left[\left(\frac{W}{T}\right)^r\right] = E[W^r]E[T^{-r}], \tag{14}$$

from which both expectations are known. □

Corollary 1 *Let $X \sim ESE\ell(0,1,\alpha,\beta;g)$. Then, it follows that*

$$E(X) = 0, \tag{15}$$

$$Var(X) = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)}\, a_1 \quad , \; for \quad \alpha > 2. \tag{16}$$

3.5. The location-scale extension

A random variable X following a location scale extended slash-elliptical distribution, which we denote by $X \sim ESE\ell(\mu,\sigma,\alpha,\beta;g)$, can be stochastically represented as

$$X = \sigma\frac{W}{T} + \mu \tag{17}$$

where $W \sim E\ell(0,1;g)$ and $T \sim Beta(\alpha,\beta)$ are independent random variables, $\alpha > 0$, $\beta > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. Some results for the location-scale are considered next. We start by presenting a general expression for the density function, which can be written as:

$$f_X(x) = \frac{1}{\sigma B(\alpha,\beta)}\int_0^1 g\left(\left(\left[\frac{x-\mu}{\sigma}\right]t\right)^2\right)t^{\alpha}(1-t)^{\beta-1}\,dt, \tag{18}$$

for $-\infty < x < \infty$.

Proposition 3 *If $X \sim ESE\ell(\mu,\sigma,\alpha,\beta;g)$ then, the r-th moment of X is given by*

$$E[X^r] = \sum_{c=1}^{n}\binom{r}{c}\sigma^c\mu^{r-c}\frac{\Gamma(\alpha-c)\Gamma(\alpha+\beta)}{\Gamma(\alpha-c+\beta)\Gamma(\alpha)}a_{c/2}, \tag{19}$$

in which

$$a_{c/2} = \int_{-\infty}^{\infty} w^c g(w^2)\,dw,\ c = 1,2,\ldots \tag{20}$$

Proof. Notice that

$$\begin{aligned}
E[X^r] &= E\left[\left(\sigma\frac{W}{T}+\mu\right)^r\right] \\
&= E\left[\sum_{c=0}^{r}\binom{r}{c}(\sigma\frac{W}{T})^c\mu^{r-c}\right] \\
&= \sum_{c=0}^{r}\binom{r}{c}\sigma^c\mu^{r-c}E[W^c]E[T^{-c}].
\end{aligned}$$

Therefore,

$$E[X^r] = \sum_{c=0}^{r}\binom{r}{c}\sigma^c\mu^{r-c}\frac{\Gamma(\alpha-c)\Gamma(\alpha+\beta)}{\Gamma(\alpha-c+\beta)\Gamma(\alpha)}a_{c/2}$$

in which

$$a_{c/2} = \int_{-\infty}^{\infty} w^c g(w^2)\,dw, \quad c = 1,2,\ldots \qquad \square$$

Corollary 2 *Let $X \sim ESE\ell(0,1,\alpha,\beta;g)$, then the kurtosis coefficient (γ_2) is given by:*

$$\gamma_2 = \frac{E\left[(X-E(X))^4\right]}{(Var(X))^2} = \frac{(\alpha-1)(\alpha-2)(\alpha+\beta-3)(\alpha+\beta-4)}{(\alpha-3)(\alpha-4)(\alpha+\beta-1)(\alpha+\beta-2)}\frac{a_2}{a_1^2}, \quad \alpha > 4. \qquad (21)$$

The kurtosis coefficient depends on the parameters α and β and, moreover, on a_1 and a_2. Tables 1 and 2 reveal that the values for the kurtosis are greater for the Student-t than for the normal distribution. Note also that for fixed β and α decreasing, the kurtosis coefficient increases, that is, the distance from the normal model gets more pronounced.

Table 1: *Kurtosis coefficients for the extended slash-elliptical for $\beta = 1$ and $\alpha > 4$ for normal and Student-t generators.*

Normal	
$a_1 = 1$,	$a_2 = 3$
α	γ_2
5	5.4
6	4.0
7	3.5714
8	3.375
9	3.2627
10	3.2

Student-t				
$a_1 = \frac{v}{v-2}$, $a_2 = \frac{3v^2}{(v-4)(v-2)}$				
α	$v = 5$	$v = 8$	$v = 20$	$v = 100$
5	16.2	8.1	6.075	5.5125
6	12	5.6	4.4999	4.0833
7	10.7143	5.3571	4.0178	3.6458
8	10.125	5.0625	3.7968	3.4453
9	9.8	4.9	3.675	3.3347
10	9.6	4.8	3.6	3.2667

Table 2: *Kurtosis coefficients for the extended slash-elliptical for $\beta = 3$ and $\alpha > 4$ for normal and Student-t generators.*

Normal	
$a_1 = 1$,	$a_2 = 3$
α	γ_2
5	8.5714
6	5.3571
7	4.3749
8	3.92
9	3.6654
10	3.5064

Student-t				
$a_1 = \frac{v}{v-2}$, $a_2 = \frac{3v^2}{(v-4)(v-2)}$				
α	$v = 5$	$v = 8$	$v = 20$	$v = 100$
5	25.7143	12.8571	9.6428	8.75
6	16.0714	8.0357	6.0268	5.4687
7	13.125	6.5624	4.9218	4.4661
8	11.76	5.88	4.41	4.0017
9	10.9963	5.498	4.1236	3.7418
10	10.5195	5.2597	3.9448	3.5795

3.6. Likelihood function

Consider a random sample of size $n, X_1, \ldots, X_n$, from the distribution $ESE\ell(\mu, \sigma, \alpha, \beta; g)$. Then the log-likelihood function for $\boldsymbol{\theta} = (\mu, \sigma, \alpha, \beta)^{\mathsf{T}}$ can be expressed as

$$\ell(\boldsymbol{\theta}; \mathbf{x}) = -n\log(\sigma) - n\log B(\alpha, \beta) + \sum_{i=1}^{n} \log(k(x_i, \boldsymbol{\theta})) \tag{22}$$

where $k(x_i, \boldsymbol{\theta}) = \int_0^1 g\left(\left(\left[\frac{x_i-\mu}{\sigma}\right] t\right)^2\right) t^{\alpha}(1-t)^{\beta-1}\, dt$.

After differentiating the log-likelihood function, the likelihood equations are given by

$$\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{x})}{\partial \mu} = \sum_{i=1}^{n} \frac{1}{k(x_i, \boldsymbol{\theta})}\, k_1(x_i, \boldsymbol{\theta}) = 0, \tag{23}$$

$$\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{x})}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{1}{k(x_i, \boldsymbol{\theta})}\, k_2(x_i, \boldsymbol{\theta}) = 0, \tag{24}$$

$$\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{x})}{\partial \alpha} = -n\{\psi(\alpha) - \psi(\alpha+\beta)\} + \sum_{i=1}^{n} \frac{1}{k(x_i, \boldsymbol{\theta})}\, k_3(x_i, \boldsymbol{\theta}) = 0, \tag{25}$$

$$\frac{\partial \ell(\boldsymbol{\theta}; \mathbf{x})}{\partial \beta} = -n\{\psi(\beta) - \psi(\alpha+\beta)\} + \sum_{i=1}^{n} \frac{1}{k(x_i, \boldsymbol{\theta})}\, k_4(x_i, \boldsymbol{\theta}) = 0. \tag{26}$$

where

$$k_1(x_i, \boldsymbol{\theta}) = \int_0^1 -\frac{2}{\sigma}\left(\frac{x_i-\mu}{\sigma}\right) t^2 g'\left(\left(\left[\frac{x_i-\mu}{\sigma}\right] t\right)^2\right) t^{\alpha}(1-t)^{\beta-1}\, dt,$$

$$k_2(x_i, \boldsymbol{\theta}) = \int_0^1 -\frac{2}{\sigma}\left(\frac{x_i-\mu}{\sigma}\right)^2 t^2 g'\left(\left(\left[\frac{x_i-\mu}{\sigma}\right] t\right)^2\right) t^{\alpha}(1-t)^{\beta-1}\, dt,$$

$$k_3(x_i, \boldsymbol{\theta}) = \int_0^1 g\left(\left(\left[\frac{x_i-\mu}{\sigma}\right] t\right)^2\right) \log(t)\, t^{\alpha}(1-t)^{\beta-1}\, dt,$$

$$k_4(x_i, \boldsymbol{\theta}) = \int_0^1 g\left(\left(\left[\frac{x_i-\mu}{\sigma}\right] t\right)^2\right) \log(1-t)\, t^{\alpha}(1-t)^{\beta-1}\, dt.$$

and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Maximum likelihood estimators (MLEs) are obtained by maximizing the above equations. No analytical solution is available for the above equations, so that iterative procedures are required.

3.7. Simulation study

As described next, a simple algorithm can be formulated to generate random deviates from the ES distribution.

(i) Simulate $W \sim N(0,1)$

(ii) Simulate $T \sim Beta(\alpha, \beta)$

(iii) Compute $X = \sigma \frac{W}{T} + \mu$

Table 3 shows results of simulations studies, illustrating the behaviour of the MLEs for 5000 generated samples of sizes n = 50, 100, 150 and 200 from distribution $ES(\mu, \sigma, \alpha, \beta)$. For each generated sample, MLEs were computed numerically using a Newton-Raphson procedure. Means and standard deviations (SD) are reported. Note that in general, as sample size increases, estimates get close to the parameter values and the empirical standard deviation (SD) gets small, as expected. Therefore, large sample properties of the maximum likelihood estimates seem to hold for moderate sample sizes.

Table 3: *Empirical means and SD for the MLE estimators of μ, σ, α and β.*

μ	σ	α	β	$\widehat{\mu}$ (SD)	$\widehat{\sigma}$ (SD)	$\widehat{\alpha}$ (SD)	$\widehat{\beta}$ (SD)
				$n = 50$			
0	1	1	2	0.2374 (0.5218)	1.0090 (0.2029)	1.2248 (0.3946)	2.8790 (1.4356)
0	1	1	5	0.4503 (0.9686)	1.0404 (0.1904)	1.1204 (0.3333)	6.3611 (3.2877)
2	10	1	1	3.1886 (3.1744)	10.5595 (1.6844)	1.3382 (0.4474)	1.6140 (0.8279)
				$n = 100$			
0	1	1	2	0.0374 (0.3259)	1.1333 (0.1613)	1.1930 (0.2213)	2.1900 (0.7484)
0	1	1	5	0.2426 (0.6733)	1.0378 (0.1297)	1.0706 (0.1768)	5.4931 (1.6113)
2	10	1	1	2.1469 (2.4254)	10.1862 (1.2163)	1.0567 (0.2685)	1.1124 (0.4572)
				$n = 150$			
0	1	1	2	0.0234 (0.2742)	1.0393 (0.1090)	1.0441 (0.1753)	2.1387 (0.5938)
0	1	1	5	0.2338 (0.5569)	1.1015 (0.1158)	1.0675 (0.1611)	5.4171 (1.3794)
2	10	1	1	2.0511 (1.6831)	10.0914 (1.0147)	1.0494 (0.1911)	1.0735 (0.3131)
				$n = 200$			
0	1	1	2	0.0023 (0.2376)	1.0366 (0.0946)	1.0389 (0.1433)	2.0983 (0.4803)
0	1	1	5	0.2307 (0.4547)	0.9951 (0.0980)	1.0267 (0.1427)	5.0110 (1.1707)
2	10	1	1	1.9983 (1.4606)	9.9764 (0.8570)	1.0262 (0.1583)	1.0382 (0.2585)

4. Numerical illustration

In the following, we present a real data application using the likelihood approach developed in the previous section. Since a numerical iterative approach is required to achieve the MLE for the $ESE\ell$, we used the function `optim` available in the `R` system. The specific method is the `L-BFGS-B` developed by Byrd et al. (1995) which allows "box constraint", that is, each variable can be given a lower and/or upper bound. This uses a limited-memory modification of the quasi-Newton method. Large sample variance estimates can be computed by inverting the Hessian matrix, which can also be computed numerically using `R`.

The data set considered is from an entomological experiment with a total of 730 ants. The ants were initially at the center of a box covered with sand and they moved toward a visual stimulus located at an angle of 180° degrees from the center of the box rounded to the nearest 10°. The data set was initially analysed in Jander (1957), and further analysed in Batschelet (1981), SenGupta and Pal (2001), Jones and Pewsey (2004) and Gómez et al. (2007).

Table 4 reveals descriptive statistics indicating the data set presents greater kurtosis than a data set typically coming from a normal distribution. Table 5 presents maximum likelihood estimates and corresponding standard deviations for normal (N), slash (S) and extended slash (ES) models. Using the Akaike information criterion (AIC) (see Akaike, 1974), it can be noticed that the extended slash (ES) model presents the smallest AIC. More strong evidence in favour the ES model is provided by the likelihood ratio statistics. Figure 2 (left side) depicts the histogram and graphical representation for estimated normal, slash and extended slash models for the ant data set. As revealed by the plots, the best fit seems the one corresponding to the ES model. Figure 2 (right side) shows the log-likelihood profile for parameter beta. Notice that the MLE is unique for the ant data.

Table 4: *Summary statistics for ant data set.*

Mean	Standard deviation	Asymmetry	Kurtosis
176.4384	62.64341	−0.2049024	4.575356

Table 5: *Parameter estimates for normal, slash and extended slash distributions.*

Parameter estimates	N(SD)	S(SD)	ES(SD)
$\widehat{\mu}$	176.438 (2.316)	181.425 (1.268)	181.321 (0.094)
$\widehat{\sigma}$	62.600 (1.638)	16.804 (1.246)	1.336 (0.108)
$\widehat{q}$	—	1.171 (0.085)	—
$\widehat{\alpha}$	—	—	1.907 (0.094)
$\widehat{\beta}$	—	—	40.084 (4.719)
Log-likelihood	−4055.670	-3972.111	−3953.321
AIC	8115.339	7950.222	7914.642

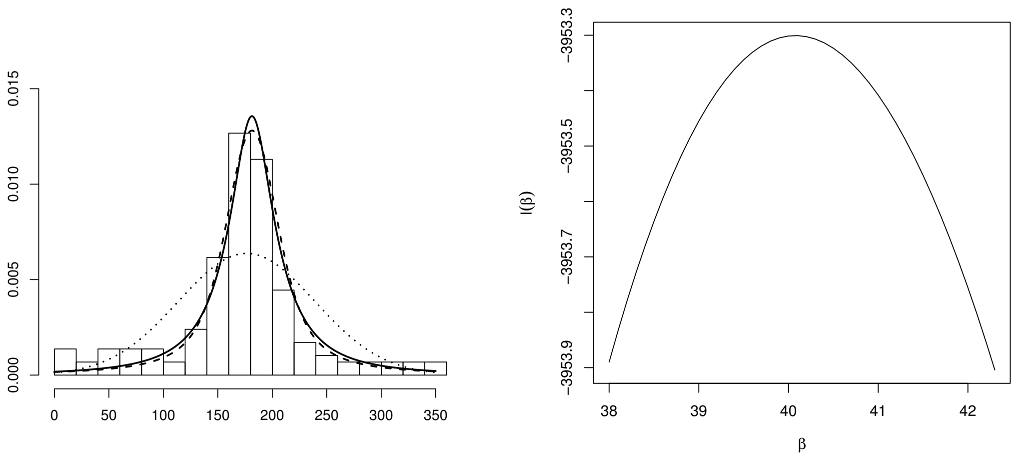

Figure 2: *Models fitted by the maximum likelihood approach for the ant direction data set: ES (solid line), S (dashed line) and N (dotted line) (left), the log-likelihood function profile of β for the ant data set (right).*

5. Multivariate case

In this section, we introduce a multivariate extended slash-elliptical model and derive some additional results concerning this extension.

The random vector $\mathbf{Y} \in \mathbb{R}^p$ follows a multivariate extended slash-elliptical distribution with location parameter $\boldsymbol{\mu}$, scale parameter matrix $\boldsymbol{\Sigma}$ (positive definite) and shape parameters $\alpha > 0$ and $\beta > 0$, which we denote by $\mathbf{Y} \sim ESEl_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha, \beta; g^{(p)})$, if

$$\mathbf{Y} = \boldsymbol{\Sigma}^{1/2}\frac{\mathbf{X}}{U} + \boldsymbol{\mu}, \tag{27}$$

where $\mathbf{X} \sim El_p(\mathbf{0}, \mathbf{I}_p; g^{(p)})$ is independent of $U \sim Beta(\alpha, \beta)$.

Proposition 4 *Let $\mathbf{Y} \sim ESEl_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha, \beta; g^{(p)})$. Then, the density of $\mathbf{Y}$ is given by*

$$h(\mathbf{y}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha, \beta) = \begin{cases} \dfrac{\boldsymbol{\Sigma}^{-1/2}}{2B(\alpha,\beta)\gamma^{(\alpha+p)/2}} \displaystyle\int_0^{\gamma} t^{\frac{\alpha+p-2}{2}} \left(1 - \frac{t^{1/2}}{\gamma^{1/2}}\right)^{\beta-1} g^{(p)}(t)dt & \mathbf{y} \neq \boldsymbol{\mu} \\ \dfrac{B(\alpha+p,\beta)}{B(\alpha,\beta)}\, \boldsymbol{\Sigma}^{-1/2} g^{(p)}(0) & \mathbf{y} = \boldsymbol{\mu} \end{cases} \tag{28}$$

where $\gamma = \| \boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})\|^2 = (\mathbf{y}-\boldsymbol{\mu})^{\top} \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})$.

Proof. Using the stochastic representation given in (27), the density function associated with $\mathbf{Y}$ is given by

$$
\begin{aligned}
h(\mathbf{y};\boldsymbol{\mu},\boldsymbol{\Sigma},\alpha,\beta) \quad &= \int_0^1 u^{\alpha+p-1} f_p(u\mathbf{y};u\boldsymbol{\mu},\boldsymbol{\Sigma}) \frac{1}{B(\alpha,\beta)}(1-u)^{\beta-1}du \\
&= \frac{1}{B(\alpha,\beta)} \int_0^1 u^{\alpha+p-1}(1-u)^{\beta-1} \boldsymbol{\Sigma}^{-1/2} g^{(p)}\left(\gamma u^2\right) du.
\end{aligned}
$$

If $\mathbf{y} = \boldsymbol{\mu}$ then the result follows straightforwardly. On the other hand, if $\mathbf{y} \neq \boldsymbol{\mu}$, after the variable change $t = (\mathbf{y}-\boldsymbol{\mu})^{\mathsf{T}} \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})u^2$, the result follows. □

Example 4 Considering $g^{(p)}(t) = \frac{1}{(2\pi)^{p/2}} e^{-t/2}$ as the generator function for the multivariate normal model and then using (28), we obtain an extension of the multivariate slash distribution introduced in Wang and Genton (2006).

Proposition 5 *Moreover, if* $\mathbf{Y} \sim ESEl_p(\boldsymbol{\mu},\boldsymbol{\Sigma},\alpha,\beta;g^{(p)})$*, then we have that*

$$
E(\mathbf{Y}) = \boldsymbol{\mu} \ \textit{ and } \ Var(\mathbf{Y}) = \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{(\alpha-1)(\alpha-2)} \alpha_g \boldsymbol{\Sigma} \ , \ \alpha > 2 \tag{29}
$$

6. Concluding remarks

This paper introduced an extension of the slash-elliptical distribution considered in Gómez et al. (2007). The distribution is called the extended slash-elliptical distribution. This new distribution is generated as the quotient between two independent random variables, one of them from the elliptical family (numerator) and the other (denominator) a beta distribution with parameters α and β. The resulting slash-elliptical distribution potentially has a larger kurtosis coefficient than the slash-elliptical distribution. We investigated properties of this distribution such as moments and closed expressions for the density function. We also derived likelihood equations for the location-scale version, placing emphasis on the special cases of the generalized slash-normal and generalized slash-Student-t models. The results of a real data application reveal that the proposed model can fit real data well, making it a viable alternative to replace models with lesser kurtosis flexibility. We also proposed a multivariate extension.

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